

SUBORDINATION BY ORTHOGONAL MARTINGALES IN $L^p, 1 < p \leq 2$

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1. INTRODUCTION: ORTHOGONAL MARTINGALES AND THE BEURLING-AHLFORS TRANSFORM

We are given two martingales on the filtration of the two dimensional Brownian motion. One is subordinated to another. We want to give an estimate of L^p -norm of a subordinated one via the same norm of a dominating one. In this setting this was done by Burkholder in [Bu1]–[Bu8]. If one of the martingales is orthogonal, the constant should drop. This was demonstrated in [BaJ1], when the orthogonality is attached to the subordinated martingale and when $2 \leq p < \infty$. This note contains an (almost obvious) observation that the same idea can be used in the case when the orthogonality is attached to a dominating martingale and $1 < p \leq 2$. Two other complementary regimes are considered in [BJVLa]. When both martingales are orthogonal, see [BJVLe]. In these two papers the constants are sharp. We are not sure of the sharpness of the constant in the present note.

A complex-valued martingale $Y = Y_1 + iY_2$ is said to be *orthogonal* if the quadratic variations of the coordinate martingales are equal and their mutual co-variation is 0:

$$\langle Y_1 \rangle = \langle Y_2 \rangle, \quad \langle Y_1, Y_2 \rangle = 0.$$

In [BaJ1], Bañuelos and Janakiraman make the observation that the martingale associated with the Beurling-Ahlfors transform is in fact an orthogonal martingale. They show that Burkholder's proof in [Bu3] naturally accommodates for this property and leads to an improvement in the estimate of $\|B\|_p$.

Theorem 1. (*One-sided orthogonality as allowed in Burkholder's proof*)

((i)) (*Left-side orthogonality*) Suppose $2 \leq p < \infty$. If Y is an orthogonal martingale and X is any martingale such that $\langle Y \rangle \leq \langle X \rangle$, then

$$\|Y\|_p \leq \sqrt{\frac{p^2 - p}{2}} \|X\|_p. \tag{1.1}$$

((ii)) (*Right-side orthogonality*) Suppose $1 < p < 2$. If X is an orthogonal martingale and Y is any martingale such that $\langle Y \rangle \leq \langle X \rangle$, then

$$\|Y\|_p \leq \sqrt{\frac{2}{p^2 - p}} \|X\|_p. \quad (1.2)$$

It is not known whether these estimates are the best possible.

Remark. The result for right-side orthogonality is stated in [JVV] and not in [BaJ1]. But [JVV] has a complicated (though funny and interesting) proof by construction a family of new Bellman functions very different from the original Burkholder's function. The goal of this small note is to demonstrate how one adapt the idea of [BaJ1] to the right-orthogonality and $1 < p \leq 2$ regime. We use just a well-known Burkholder's function here, exactly along the lines of [BaJ1].

If X and Y are the martingales associated with f and Bf respectively, then Y is orthogonal, $\langle Y \rangle \leq 4 \langle X \rangle$ and hence by (1), we obtain

$$\|Bf\|_p \leq \sqrt{2(p^2 - p)} \|f\|_p \text{ for } p \geq 2. \quad (1.3)$$

By interpolating this estimate $\sqrt{2(p^2 - p)}$ with the known $\|B\|_2 = 1$, Bañuelos and Janakiraman establish the present best estimate in publication:

$$\|B\|_p \leq 1.575(p^* - 1). \quad (1.4)$$

2. NEW QUESTIONS AND RESULTS

Since B is associated with left-side orthogonality and since we know $\|B\|_p = \|B\|_{p'}$, two important questions are

- ((i)) If $2 \leq p < \infty$, what is the best constant C_p in the left-side orthogonality problem: $\|Y\|_p \leq C_p \|X\|_p$, where Y is orthogonal and $\langle Y \rangle \leq \langle X \rangle$?
- ((ii)) Similarly, if $1 < p' < 2$, what is the best constant $C_{p'}$ in the left-side orthogonality problem?

We have separated the two questions since Burkholder's proof (and his function) already gives a good answer when $p \geq 2$. It may be (although we have now some doubts about that) the best possible as well. However no estimate (better than $p - 1$) follows from analyzing Burkholder's function when $1 < p' < 2$. Perhaps, we may hope, $C_{p'} < \sqrt{\frac{p^2 - p}{2}}$ when $1 < p' = \frac{p}{p-1} < 2$, which would then imply a better estimate for $\|B\|_p$. This paper 'answers' this hope in the negative by finding $C_{p'}$; see Theorem 2. We also ask and answer the analogous question of right-side orthogonality when $2 < p < \infty$. In the spirit of Burkholder [Bu8], we believe these

questions are of independent interest in martingale theory and may have deeper connections with other areas of mathematics.

Remark. The following sharp estimates are proved in [BJVL_a], they cover the left-side orthogonality for the regime $1 < p \leq 2$ and the right-side orthogonality for the regime $2 \leq p < \infty$. Notice that two complementary regimes have the estimates: for $2 \leq p < \infty$ and left-side orthogonality in [BaJ1], for $1 < p \leq 2$ in this note and in [JVV], but the sharpness is dubious.

Theorem 2. *Let $Y = (Y_1, Y_2)$ be an orthogonal martingale and $X = (X_1, X_2)$ be an arbitrary martingale.*

((i)) *Let $1 < p' \leq 2$. Suppose $\langle Y \rangle \leq \langle X \rangle$. Then the least constant that always works in the inequality $\|Y\|_{p'} \leq C_{p'} \|X\|_{p'}$ is*

$$C_{p'} = \frac{1}{\sqrt{2}} \frac{z_{p'}}{1 - z_{p'}} \quad (2.1)$$

where $z_{p'}$ is the least positive root in $(0, 1)$ of the bounded Laguerre function $L_{p'}$.

((ii)) *Let $2 \leq p < \infty$. Suppose $\langle X \rangle \leq \langle Y \rangle$. Then the least constant that always works in the inequality $\|X\|_p \leq C_p \|Y\|_p$ is*

$$C_p = \sqrt{2} \frac{1 - z_p}{z_p} \quad (2.2)$$

where z_p is the least positive root in $(0, 1)$ of the bounded Laguerre function L_p .

The Laguerre function L_p solves the ODE

$$sL_p''(s) + (1 - s)L_p'(s) + pL_p(s) = 0.$$

These functions are discussed further and their properties deduced in section (??); see also [?], [?], [?].

As mentioned earlier, (based however on numerical evidence) we believe in general $\sqrt{\frac{p^2 - p}{2}} < C_{p'} < p - 1$ and that these theorems cannot imply better estimates for $\|B\|_p$. However based again on numerical evidence, the following conjecture is made.

Conjecture. For $1 < p' = \frac{p}{p-1} < 2$, $C_{p'} = C_p$, or equivalently,

$$\frac{1}{\sqrt{2}} \frac{z_{p'}}{1 - z_{p'}} = \sqrt{2} \frac{1 - z_p}{z_p}.$$

It is conjecture relating the roots of the Laguerre functions. Notice that such a statement is not true with the constants from Theorem 1, and $\sqrt{\frac{2}{p'^2 - p'}} < \sqrt{\frac{p^2 - p}{2}}$ for all $p > 2$. So this conjecture (if true) suggests some distinct implications for the two settings. Note on the other hand, that the form of the two sets of constants are very analogous.

3. RIGHT-SIDE ORTHOGONALITY, $1 < p \leq 2$ REGIME, BURKHOLDER'S FUNCTION

We just repeat the approach of [BaJ1]. Let

$$\alpha_p := p \left(1 - \frac{1}{p^*} \right)^{p-1}, \quad 1 < p \leq 2.$$

For $x \in \mathbb{R}^2, y \in \mathbb{R}^2$ we define following Burkholder:

$$v(x, y) := \|y\|^p - (p^* - 1)^p \|x\|^p.$$

We consider Burkholder's function

$$u(x, y) := \alpha_p (\|y\| - (p^* - 1)\|x\|)(\|x\| + \|y\|)^{p-1}.$$

Then ($1 < p \leq 2$)

$$(p - 1)u(x, y) = -\alpha_p (\|x\| - (p - 1)\|y\|)(\|x\| + \|y\|)^{p-1}.$$

So if we denote $G(t) := u(x + ht, y + kt)$ we have

$$G''(0) = -\alpha_p(A + B + C),$$

where

$$\begin{aligned} A &= p(p - 1)(\|h\|^2 - \|k\|^2)(\|x\| + \|y\|)^{p-1}, \\ B &= (2 - p)p(\|h\|^2 - (\frac{x}{\|x\|}, h)^2)\|x\|^{-1}(\|x\| + \|y\|)^{p-1}. \end{aligned}$$

And $C \geq 0$.

Also $(p - 1)u(x, y) \leq 0$ if $\|y\| \leq \|x\|$.

Now let temporarily $X_t = (X_t^1, X_t^2), Y_t = (Y_t^1, Y_t^2)$ denote two \mathbb{R}^2 -valued martingales on the filtration of 2-Brownian motion, and let

$$d\langle X^1, X^2 \rangle = h_1^1 h_1^2 + h_2^1 h_2^2 = 0. \quad (3.1)$$

$$d\langle X^1, X^1 \rangle = (h_1^1)^2 + (h_2^1)^2 = d\langle X^2, X^2 \rangle = (h_1^2)^2 + (h_2^2)^2. \quad (3.2)$$

And let us have the following subordination by the orthogonal martingale assumption:

$$d\langle Y, Y \rangle \leq \frac{p}{2(p-1)} d\langle X, X \rangle, \quad (3.3)$$

or

$$(k_1^1)^2 + (k_2^1)^2 + (k_1^2)^2 + (k_2^2)^2 \leq \frac{p}{2(p-1)} ((h_1^1)^2 + (h_2^1)^2 + (h_1^2)^2 + (h_2^2)^2), \quad (3.4)$$

We write Itô's formula for $\mathbf{E} u(X_t, Y_t)$:

$$\mathbf{E} u(X_t, Y_t) = \mathbf{E} u(X_0, Y_0) - \frac{\alpha_p}{2} \mathbf{E} \int_0^t (A(t) + B(t) + C(t)) dt,$$

where (see above)

$$A(t) = p(p-1)(d\langle X, X \rangle_t - d\langle Y, Y \rangle_t)(\|X_t\| + \|Y_t\|)^{p-1},$$

$$B = (2-p)p(d\langle X, X \rangle_t - [(\frac{X_t}{\|X_t\|}, \vec{H}_1)^2] + (\frac{X_t}{\|X_t\|}, \vec{H}_2)^2)\|X_t\|^{-1}(\|X_t\| + \|Y_t\|)^{p-1}.$$

And $C(t) \geq 0$.

Here we denote

$$H_1 = (h_1^1, h_1^2), H_2 = (h_1^2, h_2^2),$$

or we can say that H_1 is a “vector of x stochastic derivatives of vector process X ” and H_2 is a “vector of y stochastic derivatives of vector process X ”. By (3.1) and (3.2) we get that the expression in $[\cdot]$ is

$$[(\frac{X_t}{\|X_t\|}, \vec{H}_1)^2] + (\frac{X_t}{\|X_t\|}, \vec{H}_2)^2 = \frac{1}{2} d\langle X, X \rangle.$$

Hence, if $\|Y_0\| \leq \|X_0\|$ we get (as $\|X_t\|^{-1}(\|X_t\| + \|Y_t\|)^{p-1} \geq (\|X_t\| + \|Y_t\|)^{p-2}$)

$$\mathbf{E} u(X_t, Y_t) \leq -\frac{\alpha_p}{2} \mathbf{E} \int_0^t \{p(p-1)(d\langle X, X \rangle - d\langle Y, Y \rangle) + \frac{2-p}{2(p-1)} d\langle X, X \rangle\} dt,$$

or

$$\mathbf{E} u(X_t, Y_t) \leq -\frac{\alpha_p}{2} \mathbf{E} \int_0^t \{p(p-1) \left(\frac{p}{2(p-1)} d\langle X, X \rangle - d\langle Y, Y \rangle \right)\} dt \leq 0,$$

because the integrand is positive: see the assumption of subordination (3.3). Therefore, using Burkholder's discovery that

$$v(x, y) = \|y\|^p - (p^* - 1)^p \|x\|^p \leq u(x, y)$$

we get

$$\mathbf{E} (\|Y_t\|^p - (p^* - 1)^p \|X_t\|^p) \leq \mathbf{E} u(X_t, Y_t) \leq 0,$$

and we obtain

$$\|Y\|_p \leq (p^* - 1)\|X\|_p.$$

Consider $\tilde{X} := \sqrt{\frac{p}{2(p-1)}}X$. Then (3.1) means orthogonality $d\langle \tilde{X}^1, \tilde{X}^2 \rangle = 0$. Assumption (3.2) means $d\langle \tilde{X}^1, \tilde{X}^1 \rangle = d\langle \tilde{X}^2, \tilde{X}^2 \rangle$, and (3.3) means $d\langle Y, Y \rangle \leq d\langle \tilde{X}, \tilde{X} \rangle$. Changing X to \tilde{X} we see that we proved

Theorem 3. *Let $1 < p \leq 2$, let X_t, Y_t be two martingales on the filtration of 2-dimensional Brownian motion. Let X be an orthogonal martingale, namely $d\langle X^1, X^2 \rangle = 0$ and $d\langle X^1, X^1 \rangle = d\langle X^2, X^2 \rangle$. Suppose that Y is subordinated to X :*

$$d\langle Y, Y \rangle \leq d\langle X, X \rangle.$$

Then

$$\|Y\|_p \leq \sqrt{\frac{2}{p^2 - p}}\|X\|_p.$$

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